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Ergodicity of Randomly Perturbed Lorenz Model

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We show, by using the Liapunov method, that the Lorenz model perturbed by Gaussian white noise is ergodic for any Rayleigh number. Our theory confirms two properties which have been found by numerical calculation. We also discuss the ergodicity of some other randomly perturbed dissipative systems, a one-dimensional laser, and a homopolar disk dynamo model of the geomagnetism.

KEY WORDS: Liapunov method; Lorenz model; white noise; ergodicity; dissipative system; laser; dynamo model.

1. INTRODUCTION

Influence of external random noises on dynamical systems which may become chaotic has been one of the major interests in nonequilibrium statistical mechanics.(1,2) Zippellus and Lücke(2) discussed the randomly perturbed Lorenz model:

$$\dot{x}_t = ay_t + f_t^x \tag{1a}$$

$$\dot{y}_t = (R-1)x_t - (a+1)y_t - x_t z_t + f_t^y \tag{1b}$$

$$\dot{z}_t = -bz_t + x_t y_t + x_t^2 + f_t^z \tag{1c}$$

Here x, y, and z are dimensionless variables and f_t^i $(i = x, y, z)$ is a Gaussian white noise with

$$E\left[f_t^i f_{t'}^j\right] = D\delta_{ij}\delta(t-t'), \qquad D > 0 \tag{2}$$

Parameters $a = 10$ and $b = 8/3$ are held fixed, and R is the Rayleigh number. They assumed (1) ergodicity of the system (1); their numerical

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calculation shows (2) that the long time average of F, denoted by $\langle F\rangle$

$$\langle F\rangle = \lim_{T\to\infty} T^{-1}\int_0^T F(x_t, y_t, z_t)\,dt \tag{3}$$

is independent of the starting point (x_0, y_0, z_0); and (3) that $\langle x\rangle = 0$.

The operator theory of ergodicity developed by Tropper[(3)] cannot be applied to the present case; he assumed that the drift part is given by a gradient of a certain potential function U, and that the first and second partial derivatives of grad U are bounded, both of which seem to be too restrictive in many dissipative systems.

Our discussion is based on the Liapunov method[(4,5)] for diffusion processes. We can assert the unique existence of an invariant probability measure (stationary distribution) and the ergodicity of the process if we find a Liapunov function. Main theorems obtained by the method are summarized in Section 2. An intuitive argument is also given which shows that the ergodicity has the origin in the random noise. We will establish the above facts (1), (2), (3) for any Rayleigh number R in Section 3. In many systems simple Liapunov functions can be found. As such examples we discuss the laser model[(6)] and homopolar disk dynamo model of the geomagnetism.[(7)] An application to Lotka–Volterra model of population dynamics is found in Kesten and Ogura's paper.[(8)]

2. LIAPUNOV METHOD

Consider a diffusion process (X_t) in a d-dimensional Euclidean space R^d, given by a stochastic differential equation (SDE)

$$dX_t^i = b_i(X_t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(X_t)\,dw_t^j \qquad (i = 1, 2, \ldots, d) \tag{4}$$

where $b_i(x), \sigma_{ij}(x)$ $(i, j = 1, \ldots, d)$ are $C^2(R^d)$ functions and w_t^i $(i = 1, 2, \ldots, d)$ is a d-dimensional Wiener process. Let $|\cdot|$ denote the Euclidean norm. Let τ_N be the first exit time from a ball $\{x \in R^d;\ |x| \leqslant N\}$. Let $|X_0| \leqslant N$. Then, for any given N, the SDE (4) has a unique solution (X_t) up to time τ_N. It may happen that $\lim_{N\to\infty}\tau_N < \infty$. In such a case the process is not conservative,[2] i.e., $P_x(X_t \in R^d) < 1$ for a finite t, where $P_x(\cdot)$ indicates the probability measure governing paths starting at $X_0 = x$. One such example is $d = 1$, $b_1(x) = x^2$, $\sigma_{11}(x) = 1$.[(9)]

The following proposition gives a sufficient condition for the process to be conservative in terms of a Liapunov function V.

[2] In Ref. 4 the term *regular* is used instead of *conservative*.

Proposition 1.[10] Suppose there exist a nonnegative $C^2(R^d)$ function V and a positive constant c such that

$$L_x V(x) \leqq cV(x), \qquad \forall x \in R^d \tag{5}$$

$$\lim_{N\to\infty} \inf_{|x|\geqslant N} V(x) = \infty \tag{6}$$

Then the process (X_t) is conservative, i.e., $P_x(X_t \in R^d) = 1$, $\forall t \geqslant 0$, $\forall x \in R^d$. Here L_x is a generator defined by

$$L_x = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} \tag{7}$$

with

$$a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x) \quad \blacksquare \tag{8}$$

In the following we assume that (a_{ij}) is *nondegenerate*:

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j > 0 \qquad \text{if} \quad |\xi| \neq 0, \quad \text{for all} \quad x \in R^d \tag{9}$$

As known from Ref. 11, if the process is conservative and nondegenerate, the transition probability $P(t,x,A) = P_x(X_t \in A)$ has a density $p(t,x,y)$ with respect to the Lebesgue measure dy: $P(t,x,A) = \int_A p(t,x,y)\,dy$. The density $p(t,x,y)$ is a unique classical fundamental solution[3] to the backward and forward equations

$$\begin{aligned} \partial p(t,x,y)/\partial t &= L_x p(t,x,y) \\ \partial p(t,x,y)/\partial t &= L_y^* p(t,x,y) \end{aligned} \tag{10}$$

Here L_x^* is the adjoint of L_x:

$$L_x^* \cdot = \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x) \cdot\,) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (b_i(x) \cdot\,) \tag{11}$$

The conservative diffusion process (X_t) is called *recurrent*[13] if there exists a compact K such that

$$P_x(\tau_K < \infty) = 1, \qquad \forall x \in R^d \tag{12}$$

Recurrent process (X_t) is said to be *positive recurrent*[14] if there exists a compact K such that

$$E_x[\tau_K] < \infty, \qquad \forall x \in R^d \tag{13}$$

[3] As for the existence of fundamental solutions under the assumption of $C^2(R^d)$ smoothness of $b_i(x)$, $\sigma_{ij}(x)$ and the nondegeneracy (9), see Ref. 12.

Here τ_K denotes the first hitting time of the set K: $\tau_K = \inf\{t \geqslant 0; X_t \in K\}$, and E_x denotes the expectation with respect to the probability measure P_x. The importance of the positive recurrence is given by the following.

Theorem 2.[(15),4] Suppose the process is positive recurrent. Then there exists a unique invariant probability measure μ which has a density $q(x)$ with respect to the Lebesgue measure dx. The density $q(x)$ is a unique nonnegative solution with $\int q(x)\,dx = 1$ of the stationary Fokker–Planck equation

$$L_x^* q(x) = 0 \tag{14}$$

and is given by a limit of the transition probability density p:

$$\lim_{t\to\infty} p(t,x,y) = q(y) \tag{15}$$

Furthermore the process has the ergodic properties, i.e., for any μ-integrable function F:

$$P_x\left(\lim_{T\to\infty} T^{-1}\int_0^T F(X_t)\,dt = \int F(y)\mu(dy)\right) = 1 \tag{16}$$

for all $X_0 = x \in R^d$. ■

The relation (16) shows that $\lim_{T\to\infty} T^{-1}\int_0^T F(X_t)\,dt$ does not depend on the starting point X_0.

Existence of a Liapunov function V implies the positive recurrence.

Theorem 3.[(16)] Let (X_t) be conservative. Suppose there exists a $C^2(R^d)$ nonnegative function V such that

$$\lim_{N\to\infty} \sup_{|x|\geqslant N} L_x V(x) < 0 \tag{17}$$

Then the process (X_t) is positive recurrent. ■

We sometimes need to apply the relation (16) to moment functions, i.e., $F(x) = |x|^n$. The following theorem gives a sufficient condition for both positive recurrence and μ-integrability of moment functions

Theorem 4.[(5)] Suppose there exist a $C^2(R^d)$ nonnegative function V and positive constants c_1, c_2, n and constants α, β such that

$$-\alpha + c_1|x|^n \leqslant V(x) \tag{18a}$$

$$L_x V(x) \leqslant -c_2 V(x) + \beta \tag{18b}$$

Then the process is positive recurrent and $\int |x|^n \mu(dx) < \infty$. ■

[4] Some facts used in proofs of Lemmas 9.4 and 9.5 of Ref. 15 on fundamental solutions are given in Ref. 12.

Note that the condition (18) is a stronger version of (5), (6), and (17); in fact inequality (5) is satisfied since

$$L_x(V(x) + \beta/c_2) = L_x V(x) \leqslant -c_2 V(x) + \beta \leqslant c_2(V(x) + \beta/c_2)$$

and (6) and (17) immediately follow from (18).

Models in the present paper will be solved by using Theorem 4. It is worthwhile to mention that to find Liapunov functions with (18) for diffusion processes is even easier than to find those for dynamical systems. (See Section 5.)

We also remark[5] that the existence of a Liapunov function with (18) still implies the existence of an invariant probability measure, though may not be unique, even when the diffusion coefficients (a_{ij}) become degenerate.

In the above theorems ergodicity primarily comes from the random noise, not from the drift. Existence of the Liapunov function guarantees that the attractive drift is strong enough to make the sample point in the phase space R^d hit a finite domain G around the origin. Being driven by the random noise, the sample point leaves G, and then returns to G; a trajectory drawn by the sample point is made of infinite number of excursions which are independent of each other by the Markov property. The nondegeneracy assumption (9), on the other hand, implies that the random noise remains effective in all the directions at every point in R^d, so that the trajectory will be complicated enough to smear out a d-dimensional neighborhood of each point on the trajectory. Hence the excursions will eventually smear out R^d itself, which means the ergodicity.

3. LORENZ MODEL

Equation (1) is expressed in a form of an SDE as

$$\begin{aligned} dX_t &= aY_t + \sqrt{D}\, dw_t^x \\ dY_t &= \{(R-1)X_t - (a+1)Y_t - X_t Z_t\}\, dt + \sqrt{D}\, dw_t^y \\ dZ_t &= (-bZ_t + X_t Y_t + X_t^2)\, dt + \sqrt{D}\, dw_t^z \end{aligned} \tag{19}$$

where (w_t^x, w_t^y, w_t^z) is a three-dimensional Wiener process. Generator $L_{x,y,z}$ takes the form

$$\begin{aligned} L_{x,y,z} = {} & ay\partial/\partial x + \{(R-1)x - (a+1)y - xz\}\partial/\partial y + (-bz + xy + x^2) \\ & \times \partial/\partial z + \tfrac{1}{2}D(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2) \end{aligned} \tag{20}$$

Consider a function V defined by

$$V(x, y, z) = (a+2)x^2/a + (x+y)^2 + (z - a - R)^2 \tag{21}$$

It is easy to see

$$V(x,y,z) \geqslant \alpha(x^2+y^2+z^2) - \alpha(a+R)^2/(1-\alpha) \tag{22}$$

where

$$0 < \alpha \equiv \left\{2+(a+2)/a - \left[(a+2)^2/a^2+4\right]^{1/2}\right\}/2 < 1$$

Now we see

$$\begin{aligned} L_{x,y,z}V(x,y,z) &= -2(a+1)x^2 - 2y^2 - b(z-a-R)^2 \\ &\quad - bz^2 + D(4+2/a) + b(a+R)^2 \\ &\leqslant -\tfrac{1}{2}(a\wedge 1)\{(2a+2)x^2/a + 2xy + y^2\} \\ &\quad - b(z-a-R)^2 + D(4+2/a) + b(a+R)^2 \\ &\leqslant -(\tfrac{1}{2}a \wedge b \wedge \tfrac{1}{2})V(x,y,z) + D(4+2/a) + b(a+R)^2 \end{aligned} \tag{23}$$

where $c_1 \wedge c_2 = \min(c_1, c_2)$. Hence the inequality (18) holds with $V = V(x, y, z)$. By Theorem 4 we see that the process is ergodic with respect to a unique invariant probability measure μ whose density $q(x,y,z)$ is given by a unique nonnegative solution of the stationary Fokker–Planck equation

$$L^*_{x,y,z}q(x,y,z) = 0 \tag{24}$$

and that moments up to the second order are finite:

$$\int (x^2+y^2+z^2)q(x,y,z)\,dx\,dy\,dz < \infty. \tag{25}$$

In (24) $L^*_{x,y,z}$ is the adjoint operator of $L_{x,y,z}$ (20).

Clearly Eq. (24) is invariant through the transformation $x \to -x$, $y \to -y$, $z \to z$. Hence by the uniqueness of q, we have $q(x,y,z) = q(-x, -y, z)$, which together with (25) shows

$$\begin{aligned} \int xq(x,y,z)\,dx\,dy\,dz &= \int -xq(-x,-y,z)\,dx\,dy\,dz \\ &= -\int xq(x,y,z)\,dx\,dy\,dz \end{aligned}$$

i.e., $\langle x \rangle = 0$, proving the fact (3). Similarly we obtain $\langle y \rangle = 0$.

In numerical calculation the facts (2) and (3) are confirmed only for convection regime $0 < R < R_T = 24.74$. The arguments given in this section show they hold for all the Rayleigh number $R > 0$ and $a > 0$, $b > 0$.

4. ONE-DIMENSIONAL LASER

The model is given by[6]: $d = 1$, $b_1(x) = \gamma x - x^3$, $\sigma_{11}(x) = 1$ with a constant γ. The generator L_x takes the form

$$L_x = (\gamma x - x^3) d/dx + \tfrac{1}{2} d^2/dx^2$$

A function $V(x) = x^2$ satisfies the assumption of Theorem 4; hence the process is ergodic and the density $q(x)$ of the unique invariant probability measure is given by $L_x^* q(x) = 0$, the solution of which is $q(x) = \mathcal{N} \exp(\gamma x^2 - \frac{1}{2} x^4)$ with a normalization constant $\mathcal{N}$.

5. HOMOPOLAR DISK DYNAMO MODEL

The homopolar disk dynamo model was introduced by Bullard.[7] We here discuss a slightly modified version due to Allan[17] which is given by an ordinary differential equation

$$\begin{aligned} \dot{x}_t &= -\mu x_t + x_t y_t \\ \dot{y}_t &= -\nu y_t + 1 - x_t^2 \end{aligned} \tag{26}$$

where μ, ν are positive constants with $\mu\nu < 1$. Set $\lambda = (1 - \mu\nu)^{1/2}$. The variable x corresponds to electric current, while y to the angular velocity of the disk. Stationary points of Eq. (26) are $(\lambda, \mu), (-\lambda, \mu), (0, 1/\nu)$, which cover all the possible ω-limit sets of (26). To see this, introduce a function $V_1(x, y)$ for $x > 0$ by

$$V_1(x, y) = (x^2 - \lambda^2)/2 - \lambda^2 \log(x/\lambda) + (y - \mu)^2/2 \tag{27}$$

Clearly $V_1(x, y) \geqslant 0$ and $= 0$ iff $x = \lambda$ and $y = \mu$. Using (26), we obtain

$$dV_1(x_t, y_t)/dt = -\nu(y_t - \mu)^2 \tag{28}$$

hence $dV_1/dt = 0$ iff $y = \mu$. By a theorem given in La Salle–Lefshetz' book[18], (x_t, y_t) with $x_0 > 0$ approaches to (λ, μ) as $t \to \infty$. Similarly (x_t, y_t) with $x_0 < 0$ approaches to $(-\lambda, \mu)$. If $x_0 = 0$ then $x_t \equiv 0$ and $y_t \to 1/\nu$ as $t \to \infty$.

The above discussion also shows that the model (26) does not exhibit the reversal of the geomagnetism; $x_0 x_t$ does not change sign. On the other hand, randomly perturbed homopolar disk dynamo model

$$\begin{aligned} dX_t &= -\mu X_t\, dt + X_t Y_t\, dt + \sigma\, dw_t^x \\ dY_t &= -\nu Y_t\, dt + (1 - X_t^2)\, dt + \sigma\, dw_t^y \end{aligned} \tag{29}$$

does exhibit the reversal.[19] Its ergodicity is proved by using Theorem 4 with a Liapunov function

$$V(x, y) = x^2 + y^2$$

since

$$\begin{aligned} L_{x,y} V(x, y) &= -2\mu x^2 - 2\nu y^2 + 2y + 2\sigma^2 \\ &\leqslant -(\mu \wedge \nu)(x^2 + y^2) + 1/\nu + 2\sigma^2 \end{aligned}$$

Here

$$\begin{aligned} L_{x,y} = &(-\mu x + xy)\partial/\partial x + (-\nu y + 1 - x^2)\partial/\partial y \\ &+ \tfrac{1}{2}\sigma^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) \end{aligned}$$

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